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THE RATIO SET OF THE HARMONIC MEASURE OF A RANDOM WALK ON A HYPERBOLIC GROUP

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ABSTRACT

We consider the harmonic measure on the Gromov boundary of a non-amenable hyperbolic group defined by a finite range random walk on the group, and study the corresponding orbit equivalence relation on the boundary. It is known to be always amenable and of type III. We determine its ratio set by showing that it is generated by certain values of the Martin kernel. In particular, we show that the equivalence relation is never of type III_0.

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Introduction

Given a discrete group Γ and a probability measure μ on Γ, one can construct a measure space (X,ν) with an action of Γ on it called the Poisson boundary. In many cases this space can be identified with a geometric boundary of Γ, and then the harmonic measure ν is the hitting distribution of a random walk defined by μ. E.g., if Γ is a nonelementary hyperbolic group and μ is finitely supported then the Poisson boundary coincides with the Gromov boundary [A] (see also [K1]). Conversely, a large class of measures on the Gromov boundary can be obtained as harmonic measures for certain random walks (though typically not of finite range) [CM1, CM2].

It is known that the action of Γ on (X,ν) is always ergodic, amenable and, as soon as (X,ν) is nontrivial, of type III. In particular, apart from the type III_0 case, the orbit equivalence relation is completely determined by the ratio set [CFW, Kr] . There are a number of papers dealing with the computation of the ratio sets for such actions. To name a few, Sullivan [Su1, Su2] proved that the recurrent part of an action of a discrete conformal group on the sphere S^d relative to the Lebesgue measure is of type III_1. Spatzier [Sp] showed that if Γ is the fundamental group of a compact connected negatively curved manifold M then the action of Γ on the sphere at infinity of the universal cover $\widetilde{M}$ of M considered with the visibility measure class is again of type III_1. For the case of free groups the ratio sets for certain harmonic measures were computed by Ramagge and Robertson [RR] and one of the authors [O].

The result of Spatzier mentioned above is based on the work of Bowen [B] who studied equivalence relations defined by Anosov foliations. Bowen's computations are based in turn on the fact that these equivalence relations are stably isomorphic to certain equivalence relations arising from symbolic dynamical systems with Gibbs measures. Since harmonic measures on the boundaries of free groups are Gibbs measures [La, L, Se], Bowen's results determine, in principle, the types of equivalence relations on the boundaries of free groups. Moreover, in view of connections between hyperbolic groups and symbolic dynamical systems (see e.g. [CP]), one can hope that his results can be applied to arbitrary hyperbolic groups. Indeed, following Bowen's strategy we completely determine the ratio set for any nonelementary hyperbolic group and the harmonic measure defined by a nondegenerate finite range random walk on the group. In particular, we show that the orbit equivalence relations we consider are never of type III_0.

Note that our result can be thought of as a von Neumann algebra counterpart of the results of Anantharaman-Delaroche [AD] and Laca and Spielberg [LS] saying that the crossed product C*-algebras arising from boundary actions of hyperbolic groups are purely infinite.

We cannot directly apply Bowen's result on Gibbs measures to our problem and we need to produce hyperbolic group versions of various statements he used. One of the most important ingredients of our proof is Ancona's result on almost multiplicativity of the Green kernel, which was the key observation for identification of the Martin boundary and the Gromov boundary [A]. The Hölder continuity of the Martin kernel also plays an essential role and to establish it we follow Ledrappier's argument [L] in the case of free groups. In the course of our proof, we also show a vanishing theorem for certain first cohomology of the boundary action of a hyperbolic group, which is considered as a version of Livschitz' theorem for hyperbolic dynamical systems.

1. Preliminary

We first recall the notions of Martin and Poisson boundaries, see e.g. [R] and [W] for details. Let Γ be a discrete group with a symmetric finite set S of generators. We denote by $|g|$ the word length and by $d(x,y) = |x^{-1}y|$ the word metric with respect to S. Let μ be a finitely supported probability measure on Γ. We shall assume that μ is nondegenerate in the sense that the semigroup generated by the support of μ coincides with Γ. The measure μ defines a random walk on Γ with transition probabilities $p(x,y) = \mu(x^{-1}y)$. Let P be the Markov operator with kernel $\{p(x,y)\}_{x,y}$. Assume that the random walk is transient, that is, the Green function

$$G(x,y) = \sum_{n=0}^{\infty} p^{(n)}(x,y)$$

is finite, where $\{p^{(n)}(x,y)\}_{x,y}$ is the kernel of P^n. This is the case when Γ is nonamenable. Moreover, in the latter case there exist $C > 0$ and $\lambda > 0$ such that

$$G(x,y) \le Ce^{-\lambda d(x,y)}. \tag{1.1}$$

Indeed, without loss of generality, we may assume that the support of μ is contained in S. By [DG] the spectral radius of P considered as an operator on

$\ell^2(\Gamma)$ is less than 1. Hence, there exists $z > 1$ such that the operator

$$G_z = \sum_{n=0}^{\infty} z^n P^n$$

is bounded. In particular, there exists $c > 0$ such that $(G_z\delta_y, \delta_x) \le c$. Since by assumption we have a nearest neighbor random walk, $p^{(n)}(x, y) = 0$ for $n < d(x, y)$. Hence

$$G(x, y) \le z^{-d(x,y)} \sum_{n=0}^{\infty} z^n p^{(n)}(x, y) = z^{-d(x,y)}(G_z\delta_y, \delta_x) \le z^{-d(x,y)}c,$$

which implies (1.1).

The Martin kernel is defined by

$$K(x, y) = \frac{G(x, y)}{G(e, y)},$$

where $e \in \Gamma$ is the unit element. The Martin compactification $\bar{\Gamma}$ of Γ is the smallest compactification such that $\Gamma \subset \bar{\Gamma}$ is discrete and the functions $K(x, \cdot)$, $x \in \Gamma$, extend to continuous functions on $\bar{\Gamma}$. The Martin boundary is $\partial_M\Gamma = \bar{\Gamma}\backslash\Gamma$. The left action of Γ on itself extends to a continuous action on $\bar{\Gamma}$.

Let $\Omega = \prod_{n=0}^{\infty} \Gamma$ be the path space of our random walk. For any point $g \in \Gamma$ we have a Markov measure $\mathbb{P}_g$ defined on paths starting at g. For $\mathbb{P}_g$-almost every path $\underline{x} = \{x_n\}_n \in \Omega$ the sequence $\{x_n\}_n$ converges to a point on the boundary, so we get a map $\Omega \to \partial_M\Gamma$. Denote by ν_g the image of $\mathbb{P}_g$ under this map. The measure space $(\partial_M\Gamma, \nu_e)$ can be identified with the Poisson boundary of the random walk. We shall write ν instead of ν_e. The measures ν_g, $g \in \Gamma$, are equivalent, and we have

$$K(g, \omega) = \frac{d\nu_g}{d\nu}(\omega) = \frac{dg\nu}{d\nu}(\omega),$$

where $g\nu$ is the measure defined by $g\nu(X) = \nu(g^{-1}X)$.

Next we recall basic facts about hyperbolic groups, see e.g., [GH]. As above, we assume that Γ is a finitely generated group with a symmetric finite set S of generators. Denote the closed ball with center x and radius r by

$$B(x, r) = \{g \in \Gamma \colon d(x, g) \le r\}.$$

For a subset $\Delta \subset \Gamma$, we write

$$N(\Delta, r) = \{g \in \Gamma \colon d(g, \Delta) \le r\}$$

and $d\Delta = \{g \in \Gamma \colon d(g, \Delta) = 1\}$.

The Gromov product is defined by the formula

$$(x|y)_z = \frac{1}{2}(d(x,z) + d(y,z) - d(x,y)),$$

for $x, y, z \in \Gamma$. When z is the unit element e, we simply write $(x|y) = (x|y)_e$.

Let $\delta \geq 0$. The group Γ is said to be δ-hyperbolic if

$$(x|y) \geq \min\{(x|z), (y|z)\} - \delta,$$

for every $x, y, z \in \Gamma$.

If Γ is δ-hyperbolic, then every geodesic triangle $\triangle = \{\alpha, \beta, \gamma\}$ in Γ is 4δ-slim, i.e.,

$$\alpha \subset N(\beta \cup \gamma, 4\delta), \quad \beta \subset N(\gamma \cup \alpha, 4\delta), \quad \gamma \subset N(\alpha \cup \beta, 4\delta).$$

Given a geodesic triangle $\triangle = \{\alpha, \beta, \gamma\}$, we associate to it in a natural manner a tripod, which is denoted by $T_\triangle$, and a map $f_\triangle : \triangle \to T_\triangle$ whose restriction to each geodesic segment of $\triangle$ is isometric. Then it is also known that if Γ is δ-hyperbolic, then every geodesic triangle $\triangle$ is 4δ-thin, i.e., if $f_\triangle(x) = f_\triangle(y)$, then we have $d(x, y) \leq 4\delta$.

A sequence $\{x_i\}_{i\geq 1}$ in Γ is said to converge to infinity if $\lim_{i,j\to\infty}(x_i|x_j) = \infty$. The Gromov boundary $\partial\Gamma$ is defined as the set of equivalence classes of sequences converging to infinity in Γ, where two sequences $\{x_i\}_{i\geq 1}$ and $\{y_i\}_{i\geq 1}$ are said to be equivalent if $\lim_{i,j\to\infty}(x_i|y_j) = \infty$.

The Gromov product $(p|q)$ for $p, q \in \Gamma \cup \partial\Gamma$ is defined by

$$(p|q) = \sup \liminf_{i,j\to\infty}(x_i|y_j),$$

where the sup above runs over all sequences $\{x_i\}_{i\geq 1}$ converging to p and $\{y_i\}_{i\geq 1}$ converging to q. By δ-hyperbolicity,

$$(p|q) - 2\delta \leq \liminf_{i,j\to\infty}(x_i|y_j) \leq (p|q)$$

holds for all such $\{x_i\}_{i\geq 1}$ and $\{y_i\}_{i\geq 1}$. Recall that $\Gamma \cup \partial\Gamma$ is compact equipped with the base $\{B(x,r)\} \cup \{V_r(p)\}$, where

$$V_r(p) = \{q \in \Gamma \cup \partial\Gamma \colon (q|p) > r\}.$$

One can introduce a metric ρ on $\partial\Gamma$ such that

$$a^{-(\xi|\zeta)-c} \leq \rho(\xi, \zeta) \leq a^{-(\xi|\zeta)+c},$$

for some $a > 1$ and $c \geq 0$. We call any such metric visual.

For $p, q \in \Gamma \cup \partial\Gamma$, we denote by $[p, q]$ the set of all geodesic segments (or rays, lines) between p and q. For $p, q \in \partial\Gamma$ and any geodesic rays $x \in [e, p]$ and $y \in [e, q]$, the quantity $(x(m)|y(n))$ is increasing both in m and n.

Recall next that every geodesic triangle with k vertices in $\partial\Gamma$ and $3-k$ vertices in Γ is $4(k+1)\delta$-slim.

If $\{x(n)\}_{n=0}^{\infty}$ and $\{y(n)\}_{n=0}^{\infty}$ are geodesic rays converging to the same point in $\partial\Gamma$, then for all $n \geq 0$ we have

$$d(x(n), y(n)) \leq d(x(0), y(0)) + 8\delta.$$

In addition, if $x(0) = y(0)$, then $d(x(n), y(n)) \leq 4\delta$ for $n \geq 0$.

Throughout the paper we assume that Γ is a nonelementary hyperbolic group (that is, it does not have a cyclic subgroup of finite index) and study a random walk on it defined by a finitely supported nondegenerate probability measure μ. Then the group is nonamenable, and its Martin boundary coincides with the Gromov boundary by a result of Ancona [A]. We fix a finite symmetric set S of generators containing the support of μ.

2. Multiplicativity of the Green function along geodesic segments

For any points x, y and z we have $F(x, z)G(z, y) \leq G(x, y)$, where

$$F(x, z) = \frac{G(x, z)}{G(z, z)}$$

is the probability that a path starting at x hits z. The main technical result of Ancona [A] needed to identify the Martin boundary of a hyperbolic group with its Gromov boundary is that up to a factor the converse inequality is also true if z lies on a geodesic segment $\alpha \in [x, y]$. We shall need a slightly stronger result saying that the same is true for the restriction of the random walk to any subset containing a sufficiently large neighborhood of the segment. This is essentially contained in [A], but we shall give a detailed proof for completeness.

For a subset $\Delta \subset \Gamma$ consider the induced random walk on Δ (to be precise, to get a random walk we have to add a cemetery point to Δ). We denote the corresponding quantities using the subscript Δ, so we write P_Δ, G_Δ, and so on.

PROPOSITION 2.1: *There exist $R_0 > 0$ and $C \geq 1$ such that if $x, y \in \Gamma$ and $v \in \Gamma$ lies on a geodesic segment $\alpha \in [x, y]$, then*

$$G_\Delta(x, y) \leq C F_\Delta(x, v) G_\Delta(v, y)$$

for any $\Delta \subset \Gamma$ *containing* $N(\alpha, R_0)$.

For $\Lambda \subset \Gamma$ denote by $f^{\Lambda,(n)}(x,y)$ the probability that a path from x hits Λ for the first time at the n-th step and at $y \in \Lambda$, and by $F^\Lambda(x,y) = \sum_n f^{\Lambda,(n)}(x,y)$ the probability that a path from x hits Λ for the first time at y. Dually, denote by $l^{\Lambda,(n)}(x,y)$ the probability that a path from $x \in \Lambda$ stays in $\Gamma \backslash \Lambda$ till the n-th step, when it passes through y, and put $L^\Lambda(x,y) = \sum_n l^{\Lambda,(n)}(x,y)$.

Since μ is nondegenerate, there exists $K \in \mathbb{N}$ such that $d(x,y) = 1$ implies $p^{(k)}(x,y) > 0$ for some $k \le K$. Then if f is a superharmonic function, that is, $f \ge 0$ and $Pf \le f$, we have $p^{(k)}(x,y)f(y) \le f(x)$. Thus if we set

$$C_1 = \sup\{p^{(k)}(e,g)^{-1} \colon g \in S,\ k \le K,\ p^{(k)}(e,g) > 0\},$$

we get

$$f(y) \le C_1 f(x). \tag{2.1}$$

The same is true for $x, y \in \Delta$, $d(x,y) = 1$, if f is P_Δ-superharmonic and Δ contains $N([x,y],K)$. Moreover, similar inequalities with the same constant C_1 hold for the random walk defined by the measure $\check{\mu}$, $\check{\mu}(g) = \mu(g^{-1})$. For the random walk defined by $\check{\mu}$ we write $\check{P}$, $\check{P}_\Delta$, and so on.

For $|z| < 1/\rho(P)$, where $\rho(P) = \lim_{n\to\infty} p^{(n)}(e,e)^{1/n} < 1$ is the spectral radius of our random walk, define

$$G(x,y|z) = \sum_{n=0}^{\infty} p^{(n)}(x,y) z^n.$$

We similarly introduce $F^\Lambda(x,y|z)$ and $L^\Lambda(x,y|z)$.

Observe that since $p^{(n)}_\Delta(x,y) \le p^{(n)}(x,y)$, the functions $G_\Delta(x,y|z)$, $F^\Lambda_\Delta(x,y|z)$, and so on, are well-defined for any $\Delta \subset \Gamma$ and $\Lambda \subset \Delta$.

Fix a number t such that $\rho(P) < t < 1$, and set $s = 1/t$. Next choose $\ell \in \mathbb{N}$ such that $\ell \ge 8\delta$ and $r \in \mathbb{N}$ such that $t^r C_1^{2\ell} \le 1$. Finally, put

$$R_0 = r + 2\ell + K + 1.$$

For $x, y, v \in \Gamma$ define

$$U_{x,v,y} = \{g \in \Gamma \colon (g|y)_x > d(x,v)\}.$$

Since $(g|y)_x + (g|x)_y = d(x,y)$, it follows that if v lies on a geodesic segment between x and y then $\Gamma \setminus U_{x,v,y} \subset U_{y,v,x}$.

Lemma 2.2: *There is* $C' \geq 1$ *such that*

$$G_\Delta(x,w) \leq C' F_\Delta(x,v) G_\Delta(v,w|s),$$

for $w \in \Delta \cap N(U_{x,v,y},1)$, *whenever* $x,y \in \Gamma$ *and* $v \in \Gamma$ *lies on a geodesic segment* $\alpha \in [x,y]$ *with* $N(\alpha,R_0) \subset \Delta$.

Proof. The proof follows that of [W, Proposition 27.8]. Let m be the integer part of $d(x,v)/\ell$. Consider $v_0,\dots,v_m \in \alpha$ between x and v such that $d(v_k,v) = (m-k)\ell$. We denote $U_{x,v_k,y}$ by W_k. Then we have

(i) $v_k \in N(W_k,1)$ and $W_k \subset W_{k-1}$;
(ii) if $w \in W_k$ with $d(w,v_k) > r+1+8\delta$, then $B(w,r+1) \subset W_{k-1}$.

Indeed, (i) is trivial. Let $w \in W_k$ with $d(w,v_k) > r+1+8\delta$. Since $\triangle xwy$ is 4δ-thin there is v'_k, which lies on a geodesic segment between x and w, such that $d(x,v'_k) = d(x,v_k)$ and $d(v_k,v'_k) \leq 4\delta$. Let $w' \in B(w,r+1)$. If $(w'|w)_x \leq d(x,v'_k)$, then since $\triangle xww'$ is 4δ-thin, there is v''_k, which lies on a geodesic segment between w' and w, such that $d(v''_k,v'_k) \leq 4\delta$. Then

$$d(w,v_k) \leq d(w,v''_k) + d(v''_k,v'_k) + d(v'_k,v_k) \leq r+1+8\delta,$$

which is a contradiction. Hence, $(w'|w)_x > d(x,v'_k) = d(x,v_k)$. Therefore,

$$(w'|y)_x \geq \min\{(w'|w)_x,(w|y)_x\} - \delta > d(x,v_k) - \delta \geq d(x,v_{k-1}),$$

so that $w' \in W_{k-1}$, and (ii) is proved.

The constant in the statement of the lemma will be $C' = C_1^{2r+4\ell+2}$. By induction on k we will show that

$$G_\Delta(x,w) \leq C' F_\Delta(x,v_k) G_\Delta(v_k,w|s), \tag{2.2}$$

for $w \in \Delta \cap N(W_k,1)$.

For the case $k=0$, let $w \in \Delta$. Then $d(x,v_0) < \ell$. Since Δ contains the K-neighborhood of the geodesic segment between x and v_0, by the Harnack inequality (2.1) for $G_\Delta(\cdot,w)$ we obtain

$$G_\Delta(x,w) \leq C_1^\ell G_\Delta(v_0,w) \leq C_1^\ell G_\Delta(v_0,w|s).$$

Similarly we also get

$$G_\Delta(v_0,v_0) \leq C_1^\ell G_\Delta(x,v_0)$$

and so

$$1 \leq C_1^\ell F_\Delta(x,v_0).$$

It follows that (2.2) holds for $k=0$ and all $w \in \Delta$.

Now suppose that inequality (2.2) holds for $k-1$. Using again the Harnack inequality, we have

$$G_\Delta(v_{k-1}, g|s) \le C_1^\ell G_\Delta(v_k, g|s)$$

and

$$1 \le C_1^\ell F_\Delta(v_{k-1}, v_k).$$

Therefore, it follows from the induction hypothesis that for any $g \in \Delta \cap W_{k-1}$

$$\begin{aligned} G_\Delta(x,g) &\le C' F_\Delta(x, v_{k-1}) G_\Delta(v_{k-1}, g|s) \\ &\le C' C_1^{2\ell} F_\Delta(x, v_{k-1}) F_\Delta(v_{k-1}, v_k) G_\Delta(v_k, g|s) \\ &\le C' C_1^{2\ell} F_\Delta(x, v_k) G_\Delta(v_k, g|s). \end{aligned}$$

Let $w \in \Delta \cap N(W_k, 1)$. Assume first that $d(w, v_k) \ge r+2\ell+2$. Then $d(x,w) > r$. Indeed, let $w' \in W_k$ be such that $d(w,w') \le 1$. Then $d(w', v_k) \ge r+2\ell+1$, whence $B(w', r+1) \subset W_{k-1}$ and also $B(w,r) \subset W_{k-1}$. This proves the claim since $x \notin W_{k-1}$.

It follows that any path from x to w must pass through the set

$$\Lambda = \{g \in \Delta \colon d(g,w) = r\}.$$

Therefore using that $\Lambda \subset W_{k-1}$ and $l_\Delta^{\Lambda,(n)}(g,w) = 0$ for $n < r$ we get

$$\begin{aligned} G_\Delta(x,w) &= \sum_{g \in \Lambda} G_\Delta(x,g) L_\Delta^\Lambda(g,w) \\ &\le C' C_1^{2\ell} \sum_{g \in \Lambda} F_\Delta(x, v_k) G_\Delta(v_k, g|s) t^r L_\Delta^\Lambda(g, w|s) \\ &\le C' F_\Delta(x, v_k) \sum_{g \in \Lambda} G_\Delta(v_k, g|s) L_\Delta^\Lambda(g, w|s) \\ &= C' F_\Delta(x, v_k) G_\Delta(v_k, w|s). \end{aligned}$$

Assume next that $d(v_k, w) \le r+2\ell+1$. Then Δ contains the K-neighborhood of any geodesic segment between v_k and w. Hence by applying the Harnack inequality (2.1) to the $\check{P}_\Delta$-superharmonic functions $G_\Delta(x, \cdot)$ and $G_\Delta(v_k, \cdot|s)$ we get

$$\begin{aligned} G_\Delta(x,w) &\le C_1^{r+2\ell+1} G_\Delta(x, v_k) \le C_1^{r+2\ell+1} F_\Delta(x, v_k) G_\Delta(v_k, v_k|s) \\ &\le C_1^{2r+4\ell+2} F_\Delta(x, v_k) G_\Delta(v_k, w|s). \end{aligned}$$

The proof is complete. ∎

Applying the previous lemma to $\check{\mu}$ and using that $\check{F}(g,h) = L(h,g)$, we get the following.

COROLLARY 2.3: *We have*

$$G_\Delta(w,y) \le C' G_\Delta(w,v|s) L_\Delta(v,y)$$

for $w \in \Delta \cap N(U_{y,v,x}, 1)$.

We are now ready to prove the proposition.

Proof of Proposition 2.1. We follow the proof of [W, Theorem 27.12]. For $v = x, y$ the result is obvious. Assume $v \ne x, y$. Then $x \notin U_{x,v,y}$ and $y \in U_{x,v,y}$. Since we have a nearest neighbor random walk, any path from x to y has to pass through the set $\Lambda = \Delta \cap dU_{x,v,y}$ on the way from x to y. Therefore by Lemma 2.2 we get

$$\begin{aligned} G_\Delta(x,y) &= \sum_{w\in\Lambda} G_\Delta(x,w) L^\Lambda_\Delta(w,y) \\ &\le C' F_\Delta(x,v) \sum_{w\in\Lambda} G_\Delta(v,w|s) L^\Lambda_\Delta(w,y). \end{aligned} \tag{2.3}$$

Moreover, for each $w \in \Lambda$ we have $w \in N(U_{y,v,x}, 1) \cap \Delta$. Hence, by Corollary 2.3 we also have

$$G_\Delta(w,y) \le C' G_\Delta(w,v|s) L_\Delta(v,y).$$

Recall (see, e.g., [W, Lemma 27.5]) that if f is a superharmonic function then the minimal superharmonic function dominating f on Λ is

$$f^\Lambda(g) = \sum_{h\in\Lambda} F^\Lambda(g,h) f(h).$$

Applying this to $G_\Delta(\cdot, v|s)$ we conclude that for all $a \in \Delta$

$$\begin{aligned} \sum_{w\in\Lambda} F^\Lambda_\Delta(a,w) G_\Delta(w,y) &\le C' \sum_{w\in\Lambda} F^\Lambda_\Delta(a,w) G_\Delta(w,v|s) L_\Delta(v,y) \\ &\le C' G_\Delta(a,v|s) L_\Delta(v,y), \end{aligned}$$

so that

$$\begin{aligned} \sum_{w\in\Lambda} G_\Delta(a,w) L^\Lambda_\Delta(w,y) &= \sum_{w\in\Lambda} F^\Lambda_\Delta(a,w) G_\Delta(w,y) \\ &\le C' G_\Delta(a,v|s) L_\Delta(v,y). \end{aligned} \tag{2.4}$$

Combing (2.3) and (2.4) with the resolvent equation

$$sG_\Delta(v,w|s) - G_\Delta(v,w) = (s-1)\sum_{a\in\Delta} G_\Delta(v,a|s)G_\Delta(a,w),$$

we get

$$\begin{aligned}
&G_\Delta(x,y)\\
&\le C'F_\Delta(x,v)\sum_{w\in\Lambda} G_\Delta(v,w|s)L^\Lambda_\Delta(w,y)\\
&\le C'F_\Delta(x,v)\sum_{w\in\Lambda}\left\{\frac{1}{s}G_\Delta(v,w) + (1-1/s)\sum_{a\in\Delta} G_\Delta(v,a|s)G_\Delta(a,w)\right\}L^\Lambda_\Delta(w,y)\\
&\le {C'}^2F_\Delta(x,v)\left\{\frac{1}{s}G_\Delta(v,v|s) + (1-1/s)\sum_{a\in\Delta} G_\Delta(v,a|s)G_\Delta(a,v|s)\right\}L_\Delta(v,y).
\end{aligned}$$

Choose s' such that $s < s' < 1/\rho(P)$. Then using the resolvent equation again and the fact

$$p^{(n)}_\Delta(v,v) \le p^{(n)}(v,v) \le \rho(P)^n,$$

we have

$$\begin{aligned}
&\frac{1}{s}G_\Delta(v,v|s) + (1-1/s)\sum_{a\in\Delta} G_\Delta(v,a|s)G_\Delta(a,v|s)\\
&\qquad\le \frac{1}{s}G_\Delta(v,v|s) + (1-1/s)\sum_{a\in\Delta} G_\Delta(v,a|s')G_\Delta(a,v|s)\\
&\qquad= \frac{1}{s}G_\Delta(v,v|s) + (1-1/s)\frac{1}{s'-s}\{s'G_\Delta(v,v|s') - sG_\Delta(v,v|s)\}\\
&\qquad\le \frac{1}{s}G_\Delta(v,v|s) + (1-1/s)\frac{s'}{s'-s}G_\Delta(v,v|s')\\
&\qquad\le \frac{1}{s(1-\rho(P)s)} + (1-1/s)\frac{s'}{(s'-s)(1-\rho(P)s')}.
\end{aligned}$$

Denote the last expression by C''. Then, since $G_\Delta(v,v) \ge 1$, if we set $C = {C'}^2C''$, we obtain

$$G_\Delta(x,y) \le CF_\Delta(x,v)G_\Delta(v,v)L_\Delta(v,y) = CF_\Delta(x,v)G_\Delta(v,y),$$

and the proposition is proved. ∎

It will be convenient to have a version of the above proposition for the case when v is only close to a geodesic segment.

Corollary 2.4: *For any R_1 there exists $C' \geq 1$ such that if $x, y \in \Gamma$, $\alpha \in [x, y]$ and $v \in N(\alpha, R_1)$, then*

$$G_\Delta(x, y) \leq C' F_\Delta(x, v) G_\Delta(v, y)$$

for any $\Delta \subset \Gamma$ containing $N(\alpha, R_0 + R_1)$.

Proof. By Proposition 2.1 applied to $v' \in \alpha$ such that $d(v, v') \leq R_1$ we have

$$G_\Delta(x, y) \leq C G_\Delta(x, v') G_\Delta(v', y).$$

Since $R_0 \geq K$, the set Δ contains $N([v, v'], K)$. So applying the Harnack inequality (2.1) twice we get

$$G_\Delta(x, y) \leq C C_1^{2R_1} G_\Delta(x, v) G_\Delta(v, y) = C C_1^{2R_1} F_\Delta(x, v) G_\Delta(v, v) G_\Delta(v, y),$$

so that we can take $C' = C C_1^{2R_1} G(e, e)$. ∎

3. A Harnack inequality at infinity and the Hölder condition

In [L] Ledrappier proves that in the case of a free group the Martin kernel is Hölder continuous, which is a discrete analogue of a result of Anderson and Schoen [AS]. Our goal in this section is to extend this result to hyperbolic groups.

The first step is to prove an analogue of [L, Theorem 3.1], a Harnack inequality at infinity.

We say that a function u on Γ is harmonic on $\Lambda \subset \Gamma$ if $(Pu)(g) = u(g)$ for $g \in \Lambda$. Since we consider a nearest neighbor random walk, any function which coincides with u on $N(\Lambda, 1)$ is harmonic on Λ.

For $r \geq 0$ and $p \in \Gamma$ we define

$$C_r(p) = \{g \in \Gamma \colon (g|e)_p > r\}.$$

We fix an integer R such that $R \geq R_0 + 14\delta + 2$, where R_0 is from Proposition 2.1.

Proposition 3.1: *There exists a constant $B \geq 1$ satisfying the following property. Let $m \in \mathbb{N}$, $k \in \mathbb{N}$ with $0 \leq k \leq m-1$, $p \in \Gamma$ with $|p| = 3Rm$ and $\alpha \in [p, e]$. Assume that u, v are functions on Γ which are positive and harmonic on $C_{3Rk}(p)$ and vanish at infinity on $C_{3Rk}(p)$. Then we have*

$$\frac{u(g)}{u(\alpha(3Rk + 2R))} \leq B \frac{v(g)}{v(\alpha(3Rk + 2R))} \quad \text{for } g \in C_{3R(k+1)}(p).$$

Proof. Let $T(\underline{x})$ be the first time a path $\underline{x}$ from $g \in C_{3Rk+1}(p)$ hits $\Lambda = dC_{3Rk+1}(p)$, and put $T_n = \min\{T, n\}$. Note that $\Lambda \subset C_{3Rk}(p)$. Since any path from g stays in $C_{3Rk+1}(p)$ till it hits Λ (if ever), and u is harmonic on $C_{3Rk+1}(p)$, we have

$$u(g) = \int_\Omega u(x_{T_n}) d\mathbb{P}_g(\underline{x}).$$

Since our random walk is transient, almost every path either hits Λ, or goes to infinity. Since u vanishes at infinity on $C_{3Rk}(p)$, letting $n \to \infty$ we thus get

$$u(g) = \int_{\{T<\infty\}} u(x_T) d\mathbb{P}_g(\underline{x}) = \sum_{h\in\Lambda} F^\Lambda(g,h)u(h).$$

If $g \in C_{3Rk+R}(p)$, then any path from g to $h \in \Lambda$ passes through $\Theta = dC_{3Rk+R}(p)$. Hence, denoting $C_{3Rk+1}(p)$ by Δ, we can write

$$F^\Lambda(g,h) = \sum_{a\in\Theta} \sum_{b\in\Delta\setminus\Theta} G_\Delta(g,a) L^\Theta_\Delta(a,b) p(b,h). \tag{3.1}$$

If $g \in C_{3R(k+1)}(p)$ and $a \in dC_{3Rk+R}(p)$ then $(a|e)_p > 3Rk + R - 1$, whence by δ-hyperbolicity

$$(a|g)_p \geq \min\{(a|e)_p, (e|g)_p\} - \delta > 3Rk + R - 1 - \delta.$$

Since $\triangle pga$ is 4δ-thin, we conclude that for any b lying on a geodesic segment $\gamma \in [g, a]$ we have

$$(b|g)_p > 3Rk + R - 1 - 3\delta.$$

Therefore,

$$(b|e)_p \geq \min\{(b|g)_p, (g|e)_p\} - \delta > 3Rk + R - 1 - 4\delta.$$

It follows that $N(\gamma, R_0 + 8\delta)$ is contained in $C_{3Rk+1}(p)$. On the other hand,

$$3Rk + R \geq (a|e)_p \geq \min\{(a|g)_p, (g|e)_p\} - \delta = (a|g)_p - \delta,$$

and so $(a|g)_p < 3Rk + 2R$. Hence, if $\beta \in [p, g]$, then $\beta(3Rk + 2R) \in N(\gamma, 4\delta)$. Denote $\alpha(3Rk+2R)$ by g_0. Then we also have $d(\beta(3Rk+2R), g_0) \leq 4\delta$, because $(g|e)_p > 3R(k+1)$. It follows that $g_0 \in N(\gamma, 8\delta)$. Hence, by Corollary 2.4 there exists $B \geq 1$ such that

$$F_\Delta(g, g_0)G_\Delta(g_0, a) \leq G_\Delta(g, a) \leq BF_\Delta(g, g_0)G_\Delta(g_0, a),$$

for any $g \in C_{3R(k+1)}(p)$ and $a \in \Theta$. By virtue of (3.1) we get

$$F_\Delta(g, g_0)F^\Lambda(g_0, h) \leq F^\Lambda(g, h) \leq BF_\Delta(g, g_0)F^\Lambda(g_0, h).$$

It follows that

$$F_\Delta(g,g_0)u(g_0) \le u(g) \le BF_\Delta(g,g_0)u(g_0).$$

Since the same inequalities hold for v, we get the result. ∎

Next we shall prove an analogue of [L, Lemma 3.10].

Lemma 3.2: *There exist $B' \ge 1$ and $0 \le \sigma < 1$ such that for any $m \in \mathbb{N}$, any pair $\xi, \eta \in \partial\Gamma$ with $(\xi|\eta) > 3R(m+2)$ and any geodesic ray $\{x(n)\}_{n=0}^\infty \in [e,\xi]$ the function*

$$\varphi(g) = \frac{K(g,\xi)}{K(g,\eta)}$$

has the property

$$|\varphi(g) - \varphi(h)| \le B'\sigma^{k-1}$$

for $g, h \in C_{3R(k+1)}(p)$ with $1 \le k \le m$, where $p = x(3R(m+2))$.

Proof. The functions $K(\cdot,\xi)$ and $K(\cdot,\eta)$ are positive and harmonic. We claim that they vanish at infinity on $C_{3R}(p)$. Indeed, choose a geodesic ray $\{y(n)\}_n \in [e,\eta]$. Since

$$\lim_n (x(n)|y(n)) \ge (\xi|\eta) - 2\delta$$

by δ-hyperbolicity, we have $(x(n)|y(n)) > 3R(m+2) - 2\delta$ for any sufficiently large n. Since the triangle $\triangle ey(n)x(n)$ is 4δ-thin, we then get

$$(y(n)|p) > 3R(m+2) - 4\delta. \tag{3.2}$$

Let $g \in C_{3R}(p)$. Choose $\alpha \in [p,g]$, $\beta \in [p,y(n)]$, $\gamma \in [g,y(n)]$. Denote $x(3R(m+1))$ by q. Since $(g|e)_p > 3R$, using 4δ-thinness of $\triangle gep$ we can find $a \in \alpha$ such that $d(a,q) \le 4\delta$. Since $\{\alpha,\beta,\gamma\}$ is 4δ-slim, we can find $b \in \beta \cup \gamma$ such that $d(a,b) \le 4\delta$. If $b \in \beta$ then using (3.2) and that $\triangle ey(n)p$ is 4δ-thin we get

$$(b|p) > 3R(m+2) - 6\delta,$$

whence

$$(q|p) > 3R(m+2) - 14\delta.$$

This is a contradiction as $(q|p) = 3R(m+1)$. Hence, $b \in \gamma$. It follows that $q \in N(\gamma, 8\delta)$. By Corollary 2.4 there exists a constant C such that

$$G(g,y(n)) \le CG(g,q)G(q,y(n)),$$

so that

$$K(g,\eta)=\lim_{n\to\infty}\frac{G(g,y(n))}{G(e,y(n))}\le\lim_{n\to\infty}C\frac{G(g,q)G(q,y(n))}{F(e,q)G(q,y(n))}=C\frac{G(g,q)}{F(e,q)}.$$

Since $G(\cdot,q)$ vanishes at infinity by (1.1), we conclude that $K(\cdot,\eta)$ vanishes at infinity on $C_{3R}(p)$. The same is true for $K(\cdot,\xi)$ as we could take $\eta=\xi$.

By Proposition 3.1 we have

$$\frac{K(g,\xi)}{K(x(3Rm+R),\xi)}\le B\frac{K(g,\eta)}{K(x(3Rm+R),\eta)}\quad\text{for } g\in C_{6R}(p). \tag{3.3}$$

Let $\{y(n)\}_{n=0}^{\infty}\in[e,\eta]$. Since as above $(x(n)|y(n))>3R(m+2)-2\delta$ for any n sufficiently large, we have

$$d(x(3Rm+R),y(3Rm+R))\le 4\delta.$$

Denote $x(3Rm+R)$ by a and $y(3Rm+R)$ by b. We have

$$K(a,\xi)=\lim_n\frac{G(a,x(n))}{G(e,x(n))}\le\lim_n\frac{G(a,x(n))}{F(e,a)G(a,x(n))}=\frac{1}{F(e,a)}=\frac{G(e,e)}{G(e,a)}.$$

On the other hand, by Corollary 2.4 there exists a constant C such that

$$G(e,y(n))\le CG(e,b)G(b,y(n)),$$

whence

$$K(b,\eta)\ge\lim_n\frac{G(b,y(n))}{CG(e,b)G(b,y(n))}=\frac{1}{CG(e,b)}.$$

Since $G(e,b)\le C_1^{4\delta}G(e,a)$ by (2.1), we thus get

$$K(a,\xi)/K(b,\eta)\le CC_1^{4\delta}G(e,e).$$

Put $B'=BCC_1^{4\delta}G(e,e)$. Then in view of (3.3) we have

$$0\le\varphi(g)=\frac{K(g,\xi)}{K(g,\eta)}\le B'\quad\text{for } g\in C_{6R}(p).$$

Denote $K(g,\xi)$ by $u(g)$ and $K(g,\eta)$ by $v(g)$. For $2\le k\le m+1$, define

$$\underline{\varphi}_k=\inf_{g\in C_{3Rk}(p)}\varphi(g),\quad \overline{\varphi}_k=\sup_{g\in C_{3Rk}(p)}\varphi(g)$$

and $u_k(g)=u(g)-\underline{\varphi}_k v(g)$. Note that u_k is positive on $C_{3Rk}(p)$. Applying Proposition 3.1 we have

$$D_k/B\le u_k(g)/v(g)\le D_kB\quad\text{for } g\in C_{3R(k+1)}(p),$$

where

$$D_k=\frac{u_k(x(3R(k+1)+R))}{v(x(3R(k+1)+R))}.$$

We, therefore, obtain

$$\sup_{g\in C_{3R(k+1)}}\frac{u_k(g)}{v(g)}\le B^2\inf_{g\in C_{3R(k+1)}}\frac{u_k(g)}{v(g)},$$

and so

$$\overline{\varphi}_{k+1}-\underline{\varphi}_k\le B^2(\underline{\varphi}_{k+1}-\underline{\varphi}_k).$$

Similarly by using $v_k(g)=\overline{\varphi}_k v(g)-u(g)$, we also get

$$\overline{\varphi}_k-\underline{\varphi}_{k+1}\le B^2(\overline{\varphi}_k-\overline{\varphi}_{k+1}).$$

Hence, if we set $\omega_k=\overline{\varphi}_k-\underline{\varphi}_k$, then we have

$$\omega_{k+1}+\omega_k\le B^2(\omega_k-\omega_{k+1}),$$

and thus $\omega_{k+1}\le\sigma\omega_k$, where

$$0\le\sigma=(B^2-1)/(B^2+1)<1.$$

Therefore, for $g,h\in C_{3R(k+1)}(p)$, we have

$$|\varphi(g)-\varphi(h)|\le\omega_{k+1}\le\cdots\le\sigma^{k-1}\omega_2\le B'\sigma^{k-1},$$

and the proof of the lemma is complete. ∎

We can now prove that the Martin kernel is Hölder continuous.

THEOREM 3.3: *There exist $0\le\tau<1$ and for any $g\in\Gamma$ a constant $H_g\ge 0$ such that for $\xi,\eta\in\partial\Gamma$,*

$$|K(g,\xi)-K(g,\eta)|\le H_g\tau^{(\xi|\eta)}.$$

Proof. The proof repeats that of [L, Theorem 3.7]. Let $g\in\Gamma$. Let $n\in\mathbb{N}$ be such that

$$9R+|g|\ge 3Rn>6R+|g|,$$

where R is from Proposition 3.1. Let $\xi,\eta\in\partial\Gamma$ be such that $(\xi|\eta)>3Rn$. Then

$$3R(m+3)\ge(\xi|\eta)>3R(m+2)$$

for some $m\ge n-2$. Let $\{x(n)\}_{n=0}^\infty\in[e,\xi]$. Put $p=x(3R(m+2))$. Then

$$(g|e)_p\ge|p|-|g|\ge 3Rn-|g|>6R,$$

so that $g\in C_{6R}(p)$. Take $k\in\mathbb{N}$ such that $3R(k+2)\ge(g|e)_p>3R(k+1)$. By applying Lemma 3.2 to $g,e\in C_{3R(k+1)}(p)$, we have

$$\left|\frac{K(g,\xi)}{K(g,\eta)}-1\right|\le B'\sigma^{k-1}.$$

Hence,

$$|K(g,\xi)-K(g,\eta)| \le K(g,\eta)B'\sigma^{k-1},$$

Note that

$$\begin{aligned}(\xi|\eta) &\le 3R(m+3) = |p| + 3R = (g|e)_p + (g|p)_e + 3R \\ &\le 3R(k+2) + |g| + 3R = 3R(k-1) + |g| + 12R.\end{aligned}$$

Thus if we put $\tau = \sigma^{1/3R}$ and $M_g = \sup\{K(g,\zeta)\colon \zeta \in \partial\Gamma\}$, then

$$|K(g,\xi)-K(g,\eta)| \le B'K(g,\eta)\tau^{-|g|-12R}\tau^{(\xi|\eta)} \le B'M_g\tau^{-|g|-12R}\tau^{(\xi|\eta)}. \tag{3.4}$$

for any ξ and η such that $(\xi|\eta) > 3Rn$. On the other hand, if $(\xi|\eta) \le 3Rn < 12R + |g|$ then

$$|K(g,\xi)-K(g,\eta)| \le 2M_g \le 2M_g\tau^{-|g|-12R}\tau^{(\xi|\eta)}.$$

Thus it suffices to put $H_g = 2B'M_g\tau^{-|g|-12R}$. ∎

From the proof of the above theorem we also get the following.

Corollary 3.4: *There exist $H \ge 0$ and $L_0 > 0$ such that*

$$\left|\frac{K(g,\xi)}{K(g,\eta)} - 1\right| \le H\tau^{(\xi|\eta)-|g|}$$

whenever $(\xi|\eta) - |g| \ge L_0$.

Proof. Using the notation from the proof of the theorem, take $L_0 > 9R \ge 3Rn - |g|$. Then if $(\xi|\eta) \ge L_0 + |g|$, we have $(\xi|\eta) > 3Rn$, so that by the first inequality in (3.4)

$$\left|\frac{K(g,\xi)}{K(g,\eta)} - 1\right| \le B'\tau^{-12R}\tau^{(\xi|\eta)-|g|}.$$

Thus we can take $H = B'\tau^{-12R}$. ∎

4. A Gibbs-like property of a harmonic measure

For $\xi \in \partial\Gamma$ and $R > 0$, we define $U(\xi,R)$ to be the set of all $\eta \in \partial\Gamma$ such that for any pair of geodesic rays $\{x(n)\}_{n=0}^{\infty} \in [e,\xi]$ and $\{y(n)\}_{n=0}^{\infty} \in [e,\eta]$, we have

$$\lim_{n\to\infty}(x(n)|y(n)) > R.$$

Remark that the sequence $\{(x(n)|y(n))\}_{n=0}^{\infty}$ is nondecreasing and thus the above limit always exists. These sets are considered as hyperbolic versions of cylindric sets. Note that if $\zeta \in U(\xi, R)$ then $(\zeta|\xi) > R$, and by δ-hyperbolicity if $(\zeta|\xi) > R$ then $\zeta \in U(\xi, R-2\delta)$. We also have that

$$\text{if } \eta \in U(\xi, R+\delta), \quad \text{then } U(\xi, R+\delta) \subset U(\eta, R). \tag{4.1}$$

The following property of the harmonic measure $\nu = \nu_e$ on $\partial\Gamma$ reminds of a Gibbs measure.

Theorem 4.1: *There exists $D \geq 1$ such that for every $\xi \in \partial\Gamma$ and $\{x(n)\}_{n=0}^{\infty} \in [e, \xi]$, we have*

$$\frac{1}{D} \leq \frac{\nu(U(\xi, R))}{F(e, x(R))} \leq D \quad \text{for } R \in \mathbb{N}.$$

We need the following lemma to prove the theorem.

Lemma 4.2: *Let $N \in \mathbb{N}$ be such that $N > 13\delta$. Then, for any $m \in \mathbb{N}$, $\xi \in \partial\Gamma$ and $\{x(n)\}_{n=0}^{\infty} \in [e, \xi]$, we have*

$$U(x(m)^{-1}\xi, N) \subset x(m)^{-1}U(\xi, m).$$

Proof. Let $\zeta \in x(m)U(x(m)^{-1}\xi, N)$ and $\{y(n)\}_{n=0}^{\infty} \in [x(m), \zeta]$. Then

$$\{x(m)^{-1}y(n)\}_{n=0}^{\infty} \in [e, x(m)^{-1}\zeta] \quad \text{and} \quad \{x(m)^{-1}x(m+n)\}_{n=0}^{\infty} \in [e, x(m)^{-1}\xi].$$

Since $x(m)^{-1}\zeta \in U(x(m)^{-1}\xi, N)$, we have

$$(x(m)^{-1}y(n)|x(m)^{-1}x(m+n)) > N$$

for any n sufficiently large, whence

$$4\delta \geq d(x(m)^{-1}y(N), x(m)^{-1}x(m+N)) = d(y(N), x(m+N)).$$

Now let $\{z(n)\}_{n=0}^{\infty} \in [e, \zeta]$. Since the triangle $x([0,m]) \cup y \cup z$ is 8δ-slim, there exists $a \in x([0,m]) \cup z$ such that $d(y(N), a) \leq 8\delta$. Then $d(a, x(m+N)) \leq 12\delta$. Since $N > 12\delta$, we cannot have $a \in x([0,m])$. Hence, $a \in z$. Let $a = z(k)$. As $d(z(k), x(m+N)) \leq 12\delta$, we have $|m+N-k| \leq 12\delta$, and therefore,

$$d(z(m+N), x(m+N)) \leq 24\delta.$$

It follows that for any $n \geq N+m$ we have

$$(z(n)|x(n)) \geq (z(m+N)|x(m+N)) \geq m+N-12\delta.$$

Then, for any $\{x'(n)\}_{n=0}^{\infty} \in [e, \xi]$,

$$\lim_n (z(n)|x'(n)) \geq m+N-13\delta > m.$$

Thus $\zeta \in U(\xi, m)$. ∎

Proof of Theorem 4.1. Let $\eta \in U(\xi, R)$ and $\{y(n)\}_{n=0}^{\infty} \in [e, \eta]$. Then $(x(m)|y(m)) > R$ for sufficiently large m, whence $d(x(R), y(R)) \leq 4\delta$. Thanks to Corollary 2.4 there exists a positive constant C such that

$$F(e, x(R))G(x(R), y(m)) \leq G(e, y(m)) \leq CF(e, x(R))G(x(R), y(m)).$$

Dividing by $G(e, y(m))$ and letting $m \to \infty$ we get

$$F(e, x(R))\frac{d\nu_{x(R)}}{d\nu}(\eta) \leq 1 \leq CF(e, x(R))\frac{d\nu_{x(R)}}{d\nu}(\eta).$$

Integrating over $U(\xi, R)$ we obtain

$$\nu_{x(R)}(U(\xi, R)) \leq \frac{\nu(U(\xi, R))}{F(e, x(R))} \leq C\nu_{x(R)}(U(\xi, R)) \leq C.$$

Fix $N > 13\delta$. Since

$$\nu_{x(R)}(U(\xi, R)) = \nu(x(R)^{-1}U(\xi, R)) \geq \nu(U(x(R)^{-1}\xi, N))$$

by Lemma 4.2, to prove the theorem it suffices to show that there exists a positive constant D' such that $\nu(U(x(R)^{-1}\xi, N)) \geq D'$ for any $\xi \in \partial\Gamma$, $R \in \mathbb{N}$ and $\{x(n)\}_{n=0}^{\infty} \in [e, \xi]$.

Assume such a constant does not exist. Then there exist $\xi_k \in \partial\Gamma$, $m_k \in \mathbb{N}$ and $\{x_k(n)\}_{n=0}^{\infty} \in [e, \xi_k]$ such that

$$\nu(U(x_k(m_k)^{-1}\xi_k, N)) \to 0 \quad \text{as} \quad k \to \infty.$$

Since $\partial\Gamma$ is compact, by taking a subsequence if necessary, we may assume that the sequence $\{\eta_k = x_k(m_k)^{-1}\xi_k\}_k$ converges to a point $\zeta \in \partial\Gamma$. Then (4.1) implies that for sufficiently large k the set $U(\eta_k, N)$ contains $U(\zeta, N+\delta)$. Hence, $\nu(U(\zeta, N + \delta)) = 0$. This is a contradiction, because the action of Γ on $\partial\Gamma$ is minimal and hence, any open set has positive measure. ∎

5. A Livschitz type theorem

Livschitz' theorem [KH, Theorem 19.2.1] says that every Hölder continuous cocycle of a topologically transitive hyperbolic dynamical system which vanishes on all periodic points, is a coboundary given by a Hölder continuous function with the same exponent. In this section we establish a Livschitz type theorem for the boundary action of a hyperbolic group.

It is known that every infinite order element $g \in \Gamma$ acts on $\partial\Gamma$ as a hyperbolic homeomorphism, i.e., there are exactly two fixed points g^+ and g^- in $\partial\Gamma$ such that g^+ is stable and g^- is unstable. For any open subsets $U^\pm \subset \partial\Gamma$ with $g^+ \in U^+$ and $g^- \in U^-$, it holds that $g^n(\partial\Gamma \setminus U^-) \subset U^+$ for sufficiently large $n \geq 0$. For $h \in \Gamma$, the sequence $\{g^n h\}_{n=1}^\infty$ converges to g^+ and $\{g^{-n}h\}_{n=1}^\infty$ converges to g^-.

Theorem 5.1: *Let P be a group with a two-sided invariant metric ρ such that P is complete with respect to ρ. We assume that $c\colon \Gamma \times \partial\Gamma \to P$ is a Hölder continuous cocycle, that is,*

(1) *there exist positive constants $0 < \tau < 1$ and A_g for each $g \in \Gamma$ satisfying*

$$\rho(c(g,\xi), c(g,\eta)) \leq A_g \tau^{(\xi|\eta)}, \quad \forall \xi, \eta \in \partial\Gamma,$$

(2) *the cocycle identity $c(gh,\omega) = c(g,\omega)c(h, g^{-1}\omega)$ holds for every $g, h \in \Gamma$ and $\omega \in \partial\Gamma$.*

We assume $c(g,g^+) = c(g,g^-) = e$ for every infinite order element $g \in \Gamma$. Then there exists a continuous map $b\colon \partial\Gamma \to P$ satisfying

$$c(g,\xi) = b(\xi)b(g^{-1}\xi)^{-1}, \quad \forall g \in \Gamma,\ \forall \xi \in \partial\Gamma.$$

The map b satisfies Hölder continuity with the same exponent as c, that is, there exists a positive constant A such that

$$\rho(b(\xi), b(\eta)) \leq A\tau^{(\xi|\eta)}, \quad \forall \xi, \eta \in \partial\Gamma.$$

Before proving the theorem, we show a few lemmas.

Lemma 5.2: *Let $g \in \Gamma$ be an infinite order element and $r > 0$. Then there exists a constant $C(g,r)$ depending only on g, r and δ such that the following holds for any $\xi, \eta \in \partial\Gamma \setminus V_r(g^-)$ and any natural number n:*

$$(g^n\xi|g^n\eta) \geq |g^{-n}| + (\xi|\eta) - C(g,r).$$

Proof. Let $x \in [e,\xi]$ and $y \in [e,\eta]$. Then

$$(g^n x(m)|g^n y(m)) = (x(m)|y(m)) + |g^{-n}| - (g^{-n}|x(m)) - (g^{-n}|y(m))$$

and so

$$\begin{aligned}(g^n\xi|g^n\eta) &\geq \lim_{m\to\infty}(g^n x(m)|g^n y(m)) \\ &\geq (\xi|\eta) - 2\delta + |g^{-n}| - \liminf_{m\to\infty}(g^{-n}|x(m)) - \liminf_{m\to\infty}(g^{-n}|y(m)).\end{aligned}$$

Let $z \in [e, g^-]$. Since $\{z(m)\}_{m=0}^{\infty}$ and $\{g^{-n}\}_{n=0}^{\infty}$ converge to the same point g^-, there exists m_0 such that $(g^{-n}|z(m)) > r + \delta$ holds for any $m, n \geq m_0$. Since $\xi \notin V_r(g^-)$ and $\eta \notin V_r(g^-)$, we have $(x(m)|z(m)) \leq r$ and $(y(m)|z(m)) \leq r$ for any $m \in \mathbb{N}$. Thus δ-hyperbolicity implies

$$\min\{(g^{-n}|x(m)), (g^{-n}|z(m))\} - \delta \leq (x(m)|z(m)) \leq r$$

and so $(g^{-n}|x(m)) \leq r + \delta$ holds for any $m, n \geq m_0$. In particular, for any $n \geq m_0$ we get

$$\liminf_{m \to \infty} (g^{-n}|x(m)) \leq r + \delta.$$

Therefore,

$$(g^n\xi|g^n\eta) \geq (\xi|\eta) + |g^{-n}| - 2\delta - 2\max\{r + \delta, |g^{-k}| \colon 0 \leq k < m_0\}$$

holds for any natural number n. ∎

Lemma 5.3: *Let $g \in \Gamma$ be an infinite order element and $h \in \Gamma$. We assume that $hg^+ \neq g^-$. Then there exists $m_{g,h} \in \mathbb{N}$ such that*

(1) *the element $g^n h$ is of infinite order for any $n \geq m_{g,h}$,*
(2) *the sequence $\{(g^n h)^+\}_{n=m_{g,h}}^{\infty}$ converges to g^+.*

Proof. (1). Let U be a neighborhood of g^+ such that $g^- \notin h\overline{U}$, where $\overline{U}$ is the closure of U. Then there exists $m \in \mathbb{N}$ such that $g^n hU$ is strictly included in U for any $n \geq m$. For such n, the $g^n h$-orbit of any point in $U \setminus g^n hU$ is an infinite set, which shows that $g^n h$ is of infinite order.

(2). The above argument shows that $\omega \neq (g^n h)^-$ for any $\omega \in U \setminus g^n hU$. Hence, the sequence $\{(g^n h)^k \omega\}_{k=1}^{\infty}$ converges to $(g^n h)^+$, which shows that $(g^n h)^+$ belongs to $\overline{U}$. Since this holds for every U as above and sufficiently large n, we get the statement. ∎

Lemma 5.4: *Let $g \in \Gamma$ be an infinite order element and $n \in \mathbb{N}$. For $\xi \in \partial\Gamma \setminus \{g^-\}$ we set $b_{g,n}(\xi) = c(g^{-n}, \xi)$. Then the sequence $\{b_{g,n}\}_{n=1}^{\infty}$ converges to a map $b_g \colon \partial\Gamma \setminus \{g^-\} \to P$ uniformly on every compact subset of $\partial\Gamma \setminus \{g^-\}$. Moreover, there exists a constant $C'(g, r) > 0$ such that for any $\xi, \eta \in \partial\Gamma \setminus V_r(g^-)$ the following estimate holds:*

$$\rho(b_g(\xi), b_g(\eta)) \leq C'(g, r)\tau^{(\xi|\eta)}.$$

Proof. Let $\xi, \eta \in \partial\Gamma \setminus V_r(g^-)$. By the cocycle identity, we have

$$b_{g,n+1}(\xi) = b_{g,n}(\xi)c(g^{-1}, g^n\xi).$$

Thus, thanks to Lemma 5.2, we have

$$\begin{aligned}\rho(b_{g,n}(\xi), b_{g,n+1}(\xi)) = \rho(e, c(g^{-1}, g^n\xi)) &= \rho(c(g^{-1}, g^n g^+), c(g^{-1}, g^n\xi)) \\ &\leq A_{g^{-1}}\tau^{|g^{-n}|+(g^+|\xi)-C(g,r)} \\ &\leq A_{g^{-1}}\tau^{|g^{-n}|-C(g,r)}.\end{aligned}$$

Since g is an infinite order element, there exist constants $s, t > 0$ such that $|g^{-n}| \geq sn - t$ for any $n \in \mathbb{N}$ (see [GH, Chapitre 8, Proposition 21]). Thus the first statement holds.

Since the above cocycle identity and Lemma 5.2 imply

$$\begin{aligned}\rho(b_{g,n}(\xi), b_{g,n}(\eta)) &\leq \sum_{k=0}^{n-1} \rho(c(g^{-1}, g^k\xi), c(g^{-1}, g^k\eta)) \\ &\leq \sum_{k=1}^{n-1} A_{g^{-1}}\tau^{|g^{-k}|+(\xi|\eta)-C(g,r)},\end{aligned}$$

the second statement holds with

$$C'(g,r) = \sum_{k=0}^{\infty} A_{g^{-1}}\tau^{|g^{-k}|-C(g,r)}. \qquad \blacksquare$$

Lemma 5.5: *Let $g \in \Gamma$ be an infinite order element.*

(1) *Let $h \in \Gamma$ with $hg^+ \neq g^-$. Then $b_g(hg^+) = c(h^{-1}, g^+)^{-1}$ holds.*
(2) *Let $k \in \Gamma$ and $\xi \in \partial\Gamma \setminus \{g^-, kg^-\}$. Then $b_g(\xi) = b_{kgk^{-1}}(\xi)c(k^{-1}, g^+)^{-1}$ holds.*

Proof. (1). Lemmas 5.3 and 5.4 imply

$$\begin{aligned}b_g(hg^+) &= \lim_{n\to\infty} b_{g,n}(h(g^n h)^+) = \lim_{n\to\infty} c(g^{-n}, h(g^n h)^+) \\ &= \lim_{n\to\infty} c(h^{-1}, (g^n h)^+)^{-1} c(h^{-1}g^{-n}, (g^n h)^+) \\ &= c(h^{-1}, g^+)^{-1}.\end{aligned}$$

(2). Let $l \in \Gamma$ with $lg^+ \in \partial\Gamma \setminus \{g^-, kg^-\}$. Then (1) implies

$$\begin{aligned}b_g(lg^+) = c(l^{-1}, g^+)^{-1} &= c(kl^{-1}, kg^+)^{-1} c(k^{-1}, g^+)^{-1} \\ &= c(kl^{-1}, (kgk^{-1})^+)^{-1} c(k^{-1}, g^+)^{-1} \\ &= b_{kgk^{-1}}(lk^{-1}(kgk^{-1})^+) c(k^{-1}, g^+)^{-1} \\ &= b_{kgk^{-1}}(lg^+) c(k^{-1}, g^+)^{-1}.\end{aligned}$$

Since b_g and $b_{kgk^{-1}}$ are continuous on $\partial\Gamma \setminus \{g^-, kg^-\}$ and Γg^+ is dense in $\partial\Gamma$, we get the statement. ∎

Proof of Theorem 5.1. We fix an infinite order element $g \in \Gamma$. Then Lemma 5.5 (2) with $k \in \Gamma$ satisfying $g^- \neq kg^-$ shows that b_g has a unique continuous extension $b\colon \partial\Gamma \to P$, which satisfies $b(g^-) = b_{kgk^{-1}}(g^-)c(k^{-1}, g^+)^{-1}$. Note that this value does not depend on the choice of k as above.

We first show the Hölder continuity of b. We take $h_1, h_2 \in \Gamma$ such that $\omega_0 := g^-$, $\omega_1 := h_1 g^-$ and $\omega_2 := h_2 g^-$ are distinct points. We choose $r > 0$ satisfying $V_r(\omega_i) \cap V_r(\omega_j) = \emptyset$ for any $i \neq j$ and set

$$A = \max\{C'(g, r), C'(h_1 g h_1^{-1}, r), C'(h_2 g h_2^{-1}, r)\}.$$

Note that for any $\xi \neq \eta \in \partial\Gamma$, there exists $i \in \{0, 1, 2\}$ such that $\xi, \eta \in \partial\Gamma \setminus V_r(\omega_i)$. Now Lemma 5.4 and Lemma 5.5,(2) imply

$$\rho(b(\xi), b(\eta)) \leq A\tau^{(\xi|\eta)}.$$

We claim that $b(hg^+) = c(h^{-1}, g^+)^{-1}$ holds for any $h \in \Gamma$, which has already been shown for h with $hg^+ \neq g^-$. Assume that $h \in \Gamma$ satisfies $hg^+ = g^-$. Then $g^- \neq hg^-$ and

$$b(hg^+) = b(g^-) = b_{hgh^{-1}}(g^-)c(h^{-1}, g^+)^{-1}.$$

Since we have $(hgh^{-1})^+ = g^-$, the claim follows from

$$b_{hgh^{-1}}(g^-) = b_{hgh^{-1}}((hgh^{-1})^+) = \lim_{n\to\infty} c((hgh^{-1})^{-n}, (hgh^{-1})^+) = e.$$

For any $h, k \in \Gamma$, we have

$$b(kg^+)b(h^{-1}kg^+)^{-1} = c(k^{-1}, g^+)^{-1}c(k^{-1}h, g^+) = c(h, kg^+).$$

Since $c(h, \cdot)$ and b are continuous and the Γ-orbit of g^+ is dense in $\partial\Gamma$, this finishes the proof. ∎

6. Types of harmonic measures

Let $g \in \Gamma$ be an infinite order element. We define

$$r(g) = K(g^{-1}, g^+).$$

Note that being a nonzero positive harmonic function, $K(\cdot,\xi)$ is nowhere vanishing, so that $r(g)>0$. We can also write

$$\begin{aligned} r(g) &= \lim_{n\to+\infty}\frac{G(e,g^{n+1})}{G(e,g^n)} = \lim_{n\to+\infty} G(e,g^n)^{1/n} \\ &= \lim_{n\to+\infty}\frac{F(e,g^{n+1})}{F(e,g^n)} = \lim_{n\to+\infty} F(e,g^n)^{1/n} \\ &= \sup_{n\ge1} F(e,g^n)^{1/n}, \end{aligned}$$

where the last equality follows from $F(e,g^n)F(e,g^m)=F(e,g^n)F(g^n,g^{m+n})\le F(e,g^{m+n})$. Since $F(e,g^n)\le 1$, we see that $r(g)\le 1$. We put $r(g)=1$ for any finite order element $g\in\Gamma$.

LEMMA 6.1: *The function r on Γ is a class function satisfying $r(g^k)=r(g)^k$ for $k\in\mathbb{N}$. If μ is symmetric, then $r(g)=r(g^{-1})$.*

Proof. The Martin kernel is a cocycle, that is,

$$K(gh,\xi)=K(g,\xi)K(h,g^{-1}\xi),\quad\text{and thus}\quad K(h^{-1},\xi)=K(h,h\xi)^{-1}.$$

Using that $(g^k)^+=g^+$ and $(hgh^{-1})^+=hg^+$ one easily checks that $r(g^k)=r(g)^k$ for $k\in\mathbb{N}$ and $r(hgh^{-1})=r(g)$.

When μ is symmetric, we have $G(e,g^n)=G(e,g^{-n})$. Hence, $r(g)=r(g^{-1})$. ∎

Example 6.2: Consider the simple random walk defined by the canonical symmetric generating set S of $\mathbb{F}_N$. Then

$$F(e,s)=\frac{1}{2N-1}$$

for $s\in S$, see e.g. [L, Sect. 2a]. It follows that

$$F(e,g)=(2N-1)^{-|g|}$$

for any $g\in\mathbb{F}_N$. We can then conclude that

$$r(g)=(2N-1)^{-\ell_g},$$

where ℓ_g is the minimal length of elements in the conjugacy class of g.

LEMMA 6.3: *If $g\in\Gamma$ is an infinite order element, then $r(g)<1$.*

Proof. Let $|g| = l$. By [GH, Chapitre 8, Proposition 21], there exists a (λ, c)-quasi-geodesic ray $\{f(n)\}_{n=0}^{\infty}$ on Γ such that $f(ln) = g^n$ for all $n \geq 0$. By quasi-geodesic stability [GH, Chapitre 5, Théorème 6], there exists a positive constant $H = H(\delta, \lambda, c)$ such that the quasi-geodesic segment $f([lm, ln])$ is in the H-neighborhood of some geodesic segment $\alpha_{m,n} \in [g^m, g^n]$ for $m < n$. Therefore, thanks to Corollary 2.4, there exists $C \geq 1$ such that

$$F(e, g^{m+n}) \leq CF(e, g^m)F(g^m, g^{m+n})$$

for $m, n \in \mathbb{N}$. This implies

$$F(e, g^{m(n+1)}) \leq CF(e, g^{mn})F(g^{mn}, g^{m(n+1)}) = CF(e, g^{mn})F(e, g^m).$$

Hence, we have

$$r(g^m) = \lim_{n\to\infty} \frac{F(e, g^{m(n+1)})}{F(e, g^{mn})} \leq CF(e, g^m).$$

Since $F(e, g^m) \to 0$ as $m \to \infty$ by (1.1), we see that $r(g^m) < 1$ for sufficiently large m. As $r(g^m) = r(g)^m$ by the previous lemma, it follows that $r(g) < 1$. ∎

Denote by $\mathcal{R}(\Gamma, \mu) = \mathcal{R}(\Gamma, \partial\Gamma, \nu)$ the orbit equivalence relation defined by the action of Γ on $(\partial\Gamma, \nu)$.

Recall that by definition the ratio set $r(\Gamma, \partial\Gamma, \nu)$ consists of all $\lambda \geq 0$ such that for any $\varepsilon > 0$ and any subset $A \subset \partial\Gamma$ of positive measure there exists $g \in \Gamma$ such that the set of $\omega \in gA \cap A$ satisfying

$$\left| \frac{dg\nu}{d\nu}(\omega) - \lambda \right| < \varepsilon$$

has positive measure.

Note that $\mathcal{R}(\Gamma, \mu)$ is ergodic, amenable and of type III, by [K2, Theorem 3.2.1]. Hence, $\{0, 1\} \subset r(\Gamma, \partial\Gamma, \nu)$, and $r(\Gamma, \partial\Gamma, \nu) \setminus \{0\}$ is a closed multiplicative subgroup of $(0, +\infty)$. One says that $\mathcal{R}(\Gamma, \mu)$ is of type III_0, III_λ $(0 < \lambda < 1)$ or III_1 depending on whether this group is $\{1\}$, $\{\lambda^n\}_{n\in\mathbb{Z}}$ or $(0, +\infty)$. Recall also that for $0 < \lambda \leq 1$ there is only one amenable ergodic equivalence relation of type III_λ.

We can now formulate our main result.

THEOREM 6.4: *Let Γ be a nonelementary hyperbolic group, ν the harmonic measure on $\partial\Gamma$ defined by a finitely supported nondegenerate probability measure μ on Γ. Then $r(\Gamma, \partial\Gamma, \nu) \setminus \{0\}$ is the closed multiplicative subgroup of $(0, +\infty)$ generated by $\{r(g)\}_{g\in\Gamma}$. In particular, $\mathcal{R}(\Gamma, \mu)$ is never of type III_0.*

Note that since the harmonic measure is nonatomic and any infinite order element has only two fixed points, the crossed product $L^\infty(\partial\Gamma, \nu) \rtimes \Gamma$ is a factor if Γ is torsion-free. The theorem gives then the type of this factor.

To prove the theorem we need the following lemmas.

LEMMA 6.5: *Let $g \in \Gamma$ be an infinite order element. Then there exists $L_1 > 0$ such that for any $x \in \Gamma$, there is $y \in \Gamma$ such that $|y| \le L_1$ and $|xyg^n| \ge |x| + |g^n| - L_1$ for all $n \in \mathbb{Z}$.*

Proof. Since Γ is nonelementary, there is $a \in \Gamma$ such that for $h = aga^{-1}$ we have $\{g^\pm\} \cap \{h^\pm\} = \emptyset$, see e.g., the proof of [GH, Chapitre 8, Théorème 37]. Then there is $M > 0$ such that

$$(g^n|h^m) \le M \quad \text{for all } n, m \in \mathbb{Z},$$

since otherwise we could find a subsequence $\{x_n\}_{n=1}^\infty$ of $\{g^n\}_{n\in\mathbb{Z}}$ converging to a point in $\{g^\pm\}$ and a subsequence $\{y_n\}_{n=1}^\infty$ of $\{h^n\}_{n\in\mathbb{Z}}$ converging to a point in $\{h^\pm\}$ such that $\{x_n\}_n$ and $\{y_n\}_n$ are equivalent.

Take $L_1 \ge 2M + 2\delta + 3|a|$. Let $x \in \Gamma$. If $(x^{-1}|g^n) \le M + \delta$ for any $n \in \mathbb{Z}$, then we have

$$2M + 2\delta \ge |x| + |g^n| - |xg^n|$$

and hence, we can take $y = e$. If there is $m \in \mathbb{Z}$ such that $(x^{-1}|g^m) > M + \delta$, then for any $n \in \mathbb{Z}$,

$$M \ge (h^n|g^m) \ge \min\{(x^{-1}|h^n), (x^{-1}|g^m)\} - \delta = (x^{-1}|h^n) - \delta.$$

Hence, we obtain

$$M + \delta \ge (x^{-1}|h^n) = (x^{-1}|ag^n a^{-1}).$$

Therefore, we have

$$2M + 2\delta \ge |x| + |ag^n a^{-1}| - |xag^n a^{-1}| \ge |x| + |g^n| - |xag^n| - 3|a|.$$

Thus in this case we can take $y = a$. ∎

LEMMA 6.6: *If $\{r(g)\}_{g\in\Gamma}$ is a subset of $\{\lambda^n\}_{n\in\mathbb{Z}}$ for some $0 < \lambda < 1$, then $r(\Gamma, \partial\Gamma, \nu) \setminus \{0\}$ is a subgroup of $\{\lambda^n\}_{n\in\mathbb{Z}}$.*

Proof. Let $T = -2\pi/\log\lambda$ and set $c(g,\xi) = K(g,\xi)^{\sqrt{-1}T}$. Then thanks to Corollary 3.4, the cocycle c satisfies the assumption of Theorem 5.1 with $P = \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and there exists a continuous map $b\colon \partial\Gamma \to \mathbb{T}$ satisfying $c(g,\xi) = b(\xi)b(g^{-1}\xi)^{-1}$ for all $g \in \Gamma$ and $\xi \in \partial\Gamma$. We choose a Borel map $f\colon \partial\Gamma \to (\lambda, 1]$ satisfying $b(\xi) = f(\xi)^{\sqrt{-1}T}$ for all ξ and set $\nu_1 = f\nu$. Then ν_1 is equivalent to ν and satisfies

$$\frac{dg\nu_1}{d\nu_1}(\omega) \in \{\lambda^n\}_{n\in\mathbb{Z}}, \quad \forall g \in \Gamma, \quad \forall \omega \in \partial\Gamma.$$

This shows the statement. ∎

Proof of Theorem 6.4. The proof is inspired by Bowen's computation of the ratio set of a Gibbs measure in [B, Lemma 8].

Thanks to Lemma 6.6, to prove the theorem it suffices to show that $r(g)$ belongs to the ratio set for any $g \in \Gamma$. So let $g \in \Gamma$ be an infinite order element and $\lambda = r(g) = K(g^{-1}, g^+)$. Let $\varepsilon > 0$ and $A \subset \partial\Gamma$ with $\nu(A) > 0$.

There exists a visual metric ρ on $\partial\Gamma$ such that for almost every $\omega \in A$ we have

$$\lim_{r\to 0} \frac{\nu(A \cap B(\omega, r))}{\nu(B(\omega, r))} = 1.$$

Indeed, by [BS, Theorems 9.1 and 9.2] there exists a visual metric ρ such that $(\partial\Gamma, \rho)$ embeds isometrically into $\mathbb{R}^n$ for some n. Then, the above convergence holds by a classical result of Besicovitch, see e.g. [F]. Note that using Theorem 4.1 one can then show that the convergence holds for any visual metric, but we do not need this.

Therefore, there exist $\omega \in A$, $N_0 \in \mathbb{N}$ and closed neighborhoods $W(R)$ of ω for $R \geq N_0$ such that

$$U(\omega, R - N_0) \supset W(R) \supset U(\omega, R) \tag{6.1}$$

and

$$\frac{\nu(A \cap W(R))}{\nu(W(R))} \to 1 \quad \text{as} \quad R \to \infty. \tag{6.2}$$

Fix a geodesic ray $\{w(n)\}_{n=0}^{\infty} \in [e, \omega]$.

By Corollary 3.4 there is $L_0 > 0$ such that

$$e^{-\varepsilon} \leq \frac{K(h,\xi)}{K(h,\eta)} \leq e^{\varepsilon} \quad \text{if} \quad (\xi|\eta) - |h| \geq L_0. \tag{6.3}$$

By Lemma 6.5 there exists $L_1 > 0$ such that for any x there is $y \in \Gamma$ such that

$$|y| \le L_1 \quad \text{and} \quad |xyg^n| \ge |x| + |g^n| - L_1 \quad \text{for} \ \ n \in \mathbb{Z}. \tag{6.4}$$

Since $K(g^{-1}, \cdot)$ is continuous, there is $L_2 > 0$ such that

$$\lambda e^{-\varepsilon} \le K(g^{-1}, \zeta) \le \lambda e^{\varepsilon} \quad \text{for} \ \ \zeta \in U(g^+, L_2). \tag{6.5}$$

Choose an integer $L > \max\{\delta, |g|, L_0, L_1, L_2\}$.

Fix $N \in \mathbb{N}$ to be specified later and put $x = w(N)$. Then choose y satisfying (6.4) and put $h = xy$. Consider $V = U(hg^+, N + 14L)$. We claim that

$$h^{-1}V \subset U(g^+, 11L) \quad \text{and} \quad gh^{-1}V \subset U(g^+, 10L). \tag{6.6}$$

Indeed, let $\zeta \in U(hg^+, N + 14L)$. Then $(\zeta|hg^+) > N + 14L$. Since $|h| < N + L$, we thus get

$$(h^{-1}\zeta|g^+) \ge (\zeta|hg^+) - |h| > 13L.$$

Hence, $h^{-1}\zeta \in U(g^+, 13L - 2\delta) \subset U(g^+, 11L)$. Since $|gh^{-1}| \le N + 2L$, we similarly get $gh^{-1}V \subset U(g^+, 10L)$.

Next we claim that

$$hU(g^+, 10L) \subset U(hg^+, N + 4L). \tag{6.7}$$

Let $\zeta \in U(g^+, 10L)$. Take geodesic rays $z \in [e, \zeta]$ and $v \in [e, g^+]$. Note that

$$\liminf_{n\to\infty}(z(n)|g^n) \ge \lim_{n\to\infty}(z(n)|v(n)) - \delta > 9L.$$

Since by (6.4)

$$(h^{-1}|g^n) = \frac{1}{2}(|xy| + |g^n| - |xyg^n|) \le \frac{1}{2}(|xy| + L - |x|) \le L$$

for any $n \in \mathbb{Z}$, we get

$$L \ge (h^{-1}|g^n) \ge \min\{(h^{-1}|z(n)), (z(n)|g^n)\} - \delta,$$

and so $(h^{-1}|z(n)) \le L + \delta \le 2L$ for n large enough, so that, since $|h| \ge N - L$,

$$|hz(n)| \ge |h| + |z(n)| - 4L \ge |z(n)| + N - 5L.$$

Therefore, using that $|hg^n| \ge |g^n| + N - L$ by (6.4), we get

$$\begin{aligned}(hz(n)|hg^n) &\ge \frac{1}{2}(|z(n)| + N - 5L + |g^n| + N - L - d(z(n), g^n)) \\ &= N - 3L + (z(n)|g^n) \\ &> N + 6L\end{aligned}$$

for sufficiently large n. Hence, $(h\zeta|hg^+) > N + 6L$ and thus $h\zeta \in U(hg^+, N + 6L - 2\delta) \subset U(hg^+, N + 4L)$.

By (6.6) and (6.7), we have

$$hgh^{-1}V \subset U(hg^+, N + 4L). \tag{6.8}$$

For any $\zeta \in V$, by using the cocycle property we obtain

$$K(hg^{-1}h^{-1}, \zeta) = K(g^{-1}, h^{-1}\zeta)\frac{K(h, \zeta)}{K(h, hgh^{-1}\zeta)}.$$

Since $h^{-1}\zeta \in U(g^+, 11L)$ by (6.6), the first factor on the right hand side of the above equality is in $[\lambda e^{-\varepsilon}, \lambda e^{\varepsilon}]$ by (6.5). Since $|h| \leq N + L$, we have, by (6.8),

$$\begin{aligned}(\zeta|hgh^{-1}\zeta) &\geq \min\{(\zeta|hg^+), (hgh^{-1}\zeta|hg^+)\} - 2\delta > N + 4L - 2\delta \\ &\geq N + 2L \\ &\geq |h| + L.\end{aligned}$$

Hence, the second factor is in $[e^{-\varepsilon}, e^{\varepsilon}]$ by (6.3). Thus

$$\lambda e^{-2\varepsilon} \leq K(hg^{-1}h^{-1}, \zeta) \leq \lambda e^{2\varepsilon} \quad \text{for} \quad \zeta \in V. \tag{6.9}$$

To complete the proof it suffices to show that by choosing sufficiently large N we could arrange $\nu(V \cap A \cap hg^{-1}h^{-1}A) > 0$.

We shall check first that

$$V \subset U(\omega, N - 4L) \quad \text{and} \quad hgh^{-1}V \subset U(\omega, N - 4L). \tag{6.10}$$

By virtue of (6.8) it suffices to show that $U(hg^+, N + 4L) \subset U(\omega, N - 4L)$. Since

$$\begin{aligned}2(w(n)|hg^n) &\geq n + |hg^n| - |w(n)^{-1}w(N)| - |w(N)^{-1}hg^n| \\ &\geq n + (N + |g^n| - L) - (n - N) - (L + |g^n|) \\ &= 2N - 2L,\end{aligned}$$

we have $(\omega|hg^+) \geq N - L$, so that $\omega \in U(hg^+, N - 3L) \subset U(hg^+, N - 4L + \delta)$. By (4.1), it follows that $U(hg^+, N-3L) \subset U(hg^+, N-4L+\delta) \subset U(\omega, N-4L)$, and (6.10) is proved. Similarly we have $U(\omega, N - 3L) \subset U(hg^+, N - 4L)$.

Then by Theorem 4.1, for $N > 4L$ and $\{v(n)\}_{n=0}^{\infty} \in [e, hg^+]$ we obtain

$$\begin{aligned}\nu(V) &\geq \frac{F(e, v(N+14L))}{D} \\ &\geq \frac{F(e, v(N-4L))F(v(N-4L), v(N+14L))}{D} \\ &= \frac{F(e, v(N-4L))F(e, v(N-4L)^{-1}v(N+14L))}{D} \\ &\geq \frac{\nu(U(hg^+, N-4L))F(e, v(N-4L)^{-1}v(N+14L))}{D^2} \\ &\geq \frac{\nu(U(\omega, N-3L))F(e, v(N-4L)^{-1}v(N+14L))}{D^2}\end{aligned}$$

and similarly

$$\nu(U(\omega, N-3L)) \geq \frac{\nu(U(\omega, N-N_0-4L))F(e, w(N-N_0-4L)^{-1}w(N-3L))}{D^2}.$$

Thus if we put

$$c = \min\left\{\frac{F(e, x_1)F(e, x_2)}{D^4} \colon |x_1| = 18L, |x_2| = N_0 + L\right\} > 0,$$

then

$$\nu(V) \geq c\nu(U(\omega, N-N_0-4L)) \geq c\nu(W(N-4L)),$$

where the second inequality follows from (6.1). Let $\varepsilon_N \geq 0$ be such that

$$\nu(A \cap W(N-4L)) = (1-\varepsilon_N)\nu(W(N-4L)).$$

Then $\varepsilon_N \to 0$ as $N \to \infty$ by (6.2). So if we denote $A \cap V$ by X, then since $V \subset W(N-4L)$ by (6.1) and (6.10), we get

$$\begin{aligned}\nu(X) &\geq \nu(A \cap W(N-4L)) - \nu(W(N-4L) \setminus V) \\ &= (1-\varepsilon_N)\nu(W(N-4L)) - (\nu(W(N-4L)) - \nu(V)) \\ &= -\varepsilon_N\nu(W(N-4L)) + \nu(V) \\ &\geq (c-\varepsilon_N)\nu(W(N-4L)) > 0\end{aligned}$$

for N sufficiently large. Since

$$\begin{aligned}\nu(hgh^{-1}X) = \int_X K(hg^{-1}h^{-1}, \zeta)d\nu(\zeta) &\geq \lambda e^{-2\varepsilon}\nu(X) \\ &\geq (c-\varepsilon_N)\lambda e^{-2\varepsilon}\nu(W(N-4L)),\end{aligned}$$

we similarly obtain

$$\nu((hgh^{-1}X) \cap A) \geq \{(c-\varepsilon_N)\lambda e^{-2\varepsilon} - \varepsilon_N\}\nu(W(N-4L)).$$

The latter expression is strictly positive if N is sufficiently large. Thus $\nu(V \cap A \cap hg^{-1}h^{-1}A) = \nu(X \cap hg^{-1}h^{-1}A) > 0$, and the proof of the theorem is complete. ∎

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